Instrumentation and Information Processing

Chapter 4: Orthogonality in measurement

Hiroyuki Shinoda

http://www.hapis.k.u-tokyo.ac.jp/

hiroyuki_shinoda@k.u-tokyo.ac.jp

Conclusion of this chapter

Quantity to be measured: $\boldsymbol{x}$, Noise : $\boldsymbol{w}$,
Sensor output: $\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{w}$

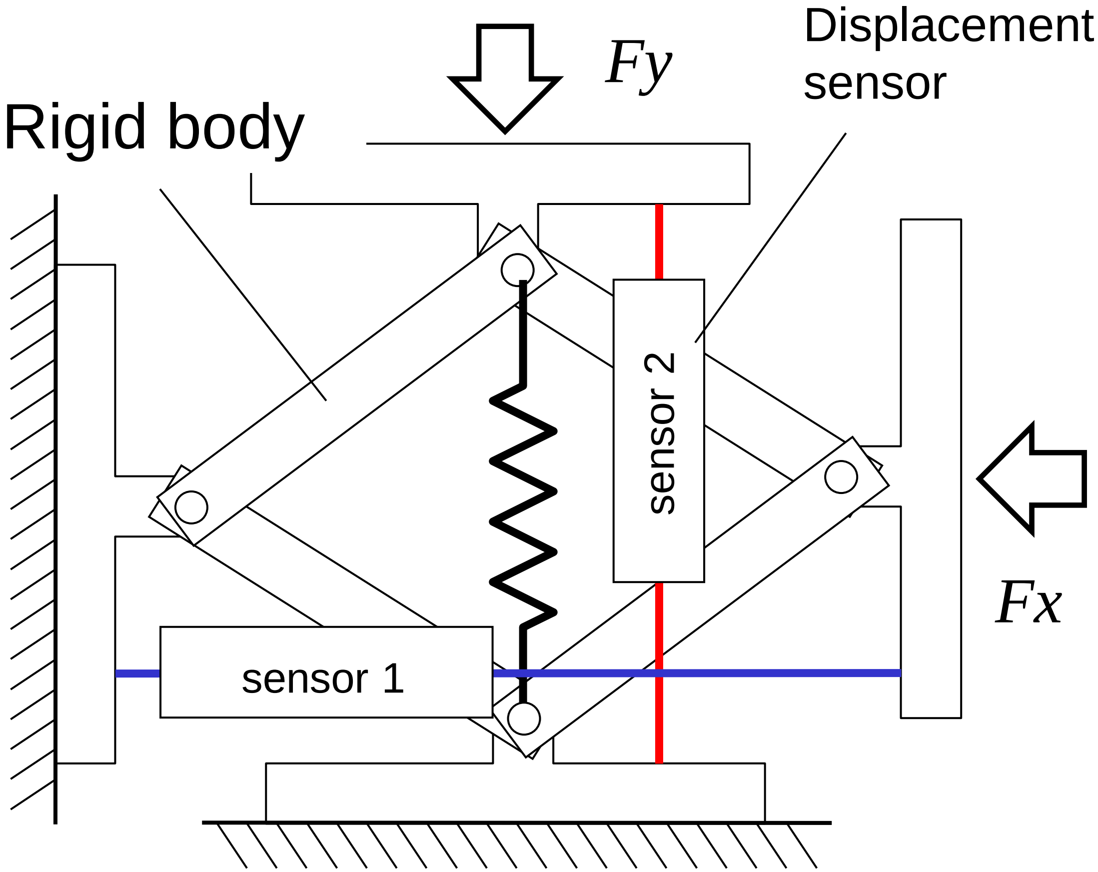
∗ Each component of $\boldsymbol{w}$ is random and equal in variance.

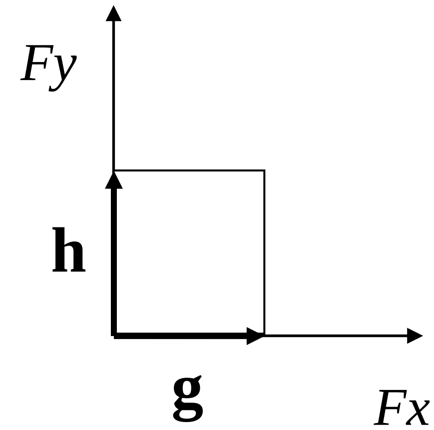
In the above measurement system,
the ratio of the largest singular value to the smallest one of matrix A is close to 1.

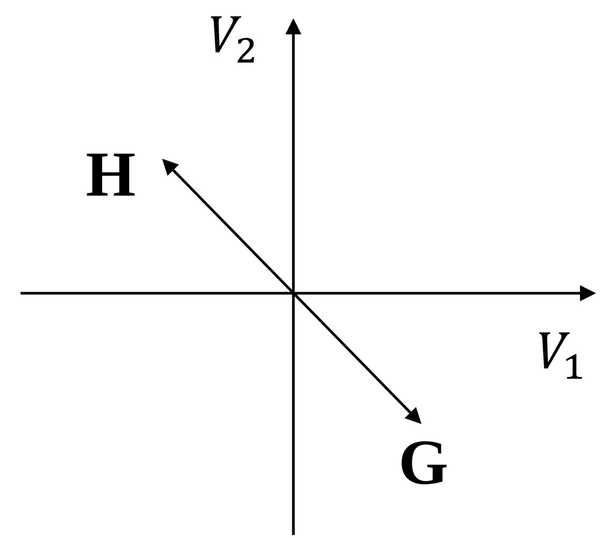
$$\Updownarrow$$

Each component of $\boldsymbol{x}$ can be measured with equal accuracy.

Nonsense two-dimensional sensor







It is impossible to estimate $\left(F_x, F_y\right)$ from the displacement sensor output (V_1, V_2).

(F_x, F_y) can not be identified if $\boldsymbol{G} = -\boldsymbol{H}$.

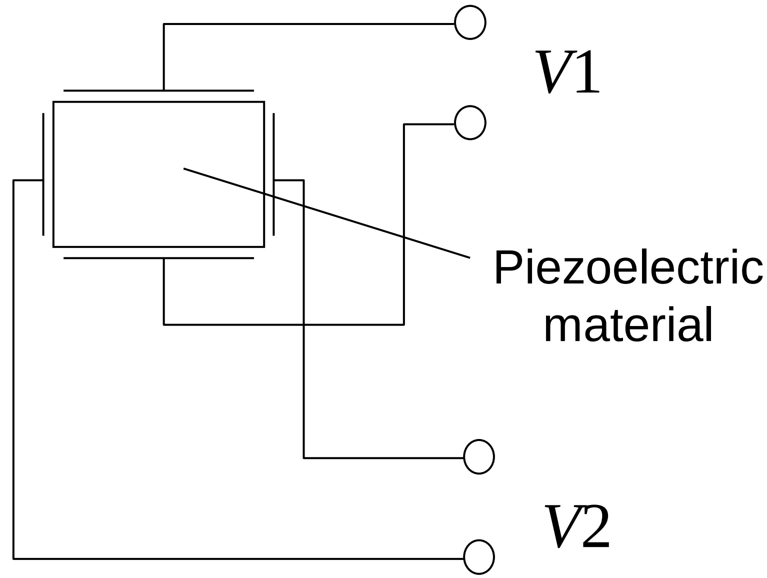
- But partial information can be obtained
- What component is measurable?

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} Fx + Fy \\ Fx - Fy \end{pmatrix} \Leftrightarrow \begin{pmatrix} Fx \\ Fy \end{pmatrix} = \frac{\alpha}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

α: Unmeasurable
β: Measurable

Orthogonality in measurement

Example: Stress sensor



Quantity to be measured
--- Stress (S_{xx}, S_{yy})

Sensor output
--- Voltage ($V1$, $V2$)

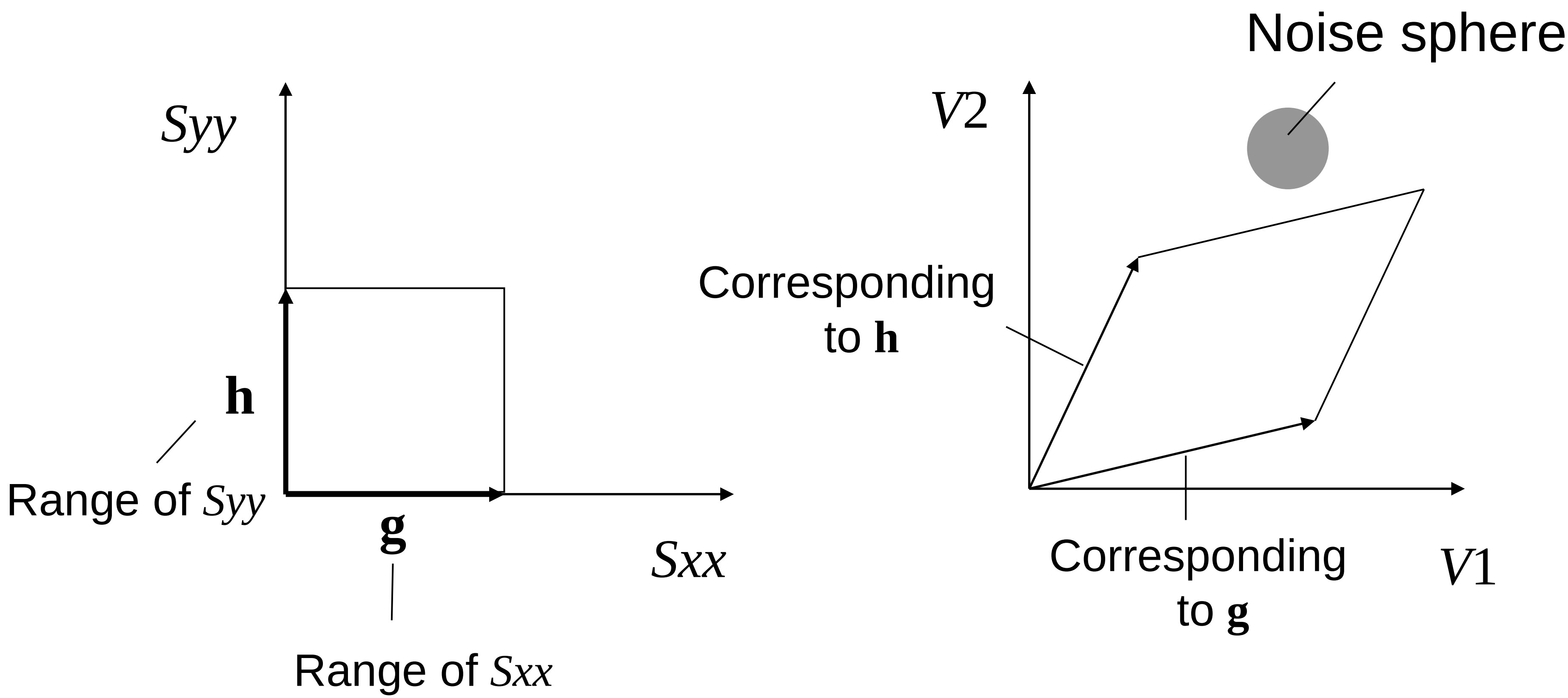
True value Noise

$$\begin{pmatrix} S_{xx} \\ S_{yy} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} V1 \\ V2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} e1 \\ e2 \end{pmatrix}$$

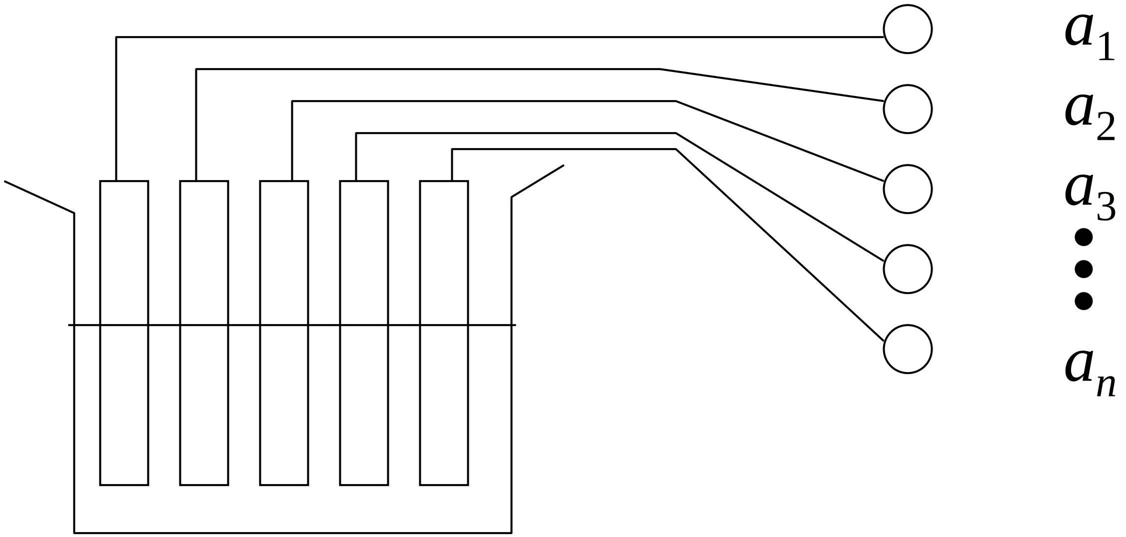
$$\begin{pmatrix} S_{xx} \\ S_{yy} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} V1 \\ V2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} e3 \\ e4 \end{pmatrix}$$

Quantity to be obtained $\left(s_{xx}, s_{yy}\right)$

Sensor output (V_1, V_2)



Example 2: Chemical sensor

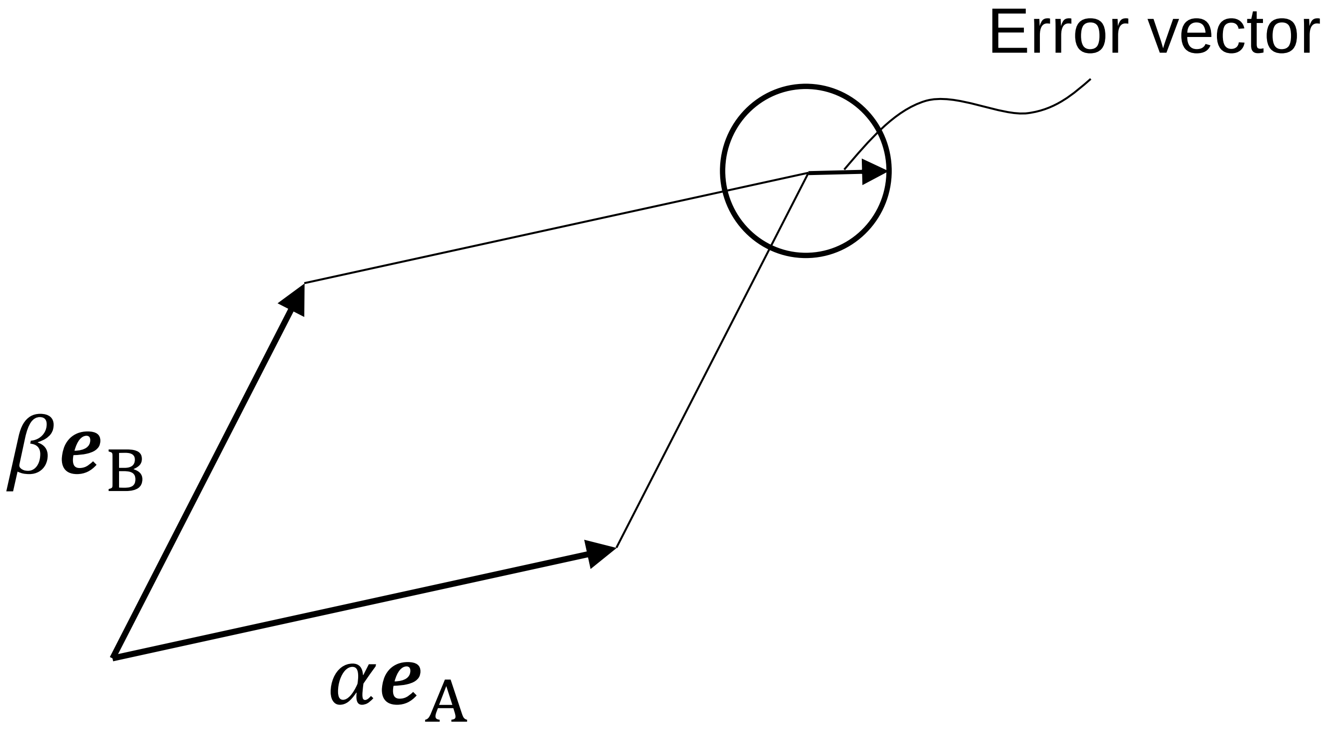


Sensor output for

component A of concentration α ⟶ $\alpha \mathbf{e}_\mathrm{A}, \quad \mathbf{e}_\mathrm{A} = (a_1, a_2, \cdots a_n)$

component B of concentration β ⟶ $\beta \mathbf{e}_\mathrm{B}, \quad \mathbf{e}_\mathrm{B} = (b_1, b_2, \cdots b_n)$

Measuring components (α, β) simultaneously



Question

Assume that an organism performs a kind of memory operation by autonomously changing and holding the concentrations of three chemical components (x_1, x_2, x_3) [%] in the body fluid. In the body, there are three kinds of sensors that output as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and the sensory system can detect each component of (y_1, y_2, y_3) with an error of about $\pm$ 0.1. Assume that x_i can take values in the range of $0 < x_i < 10$, and there is no correlation among the sensing errors of each component.

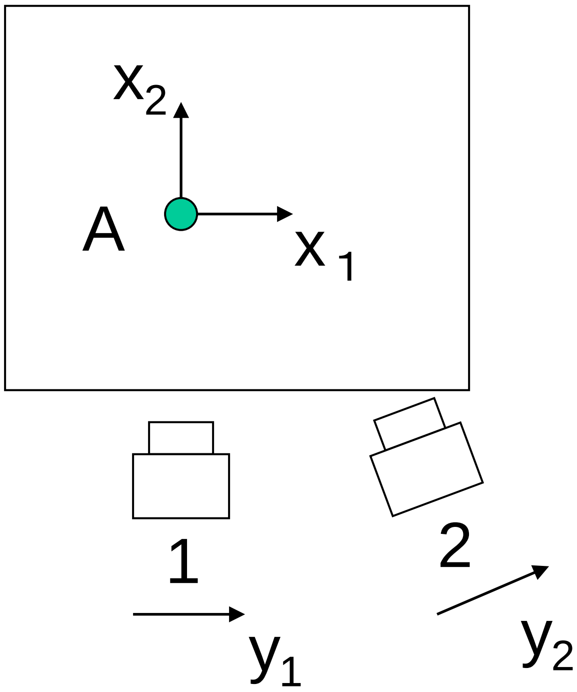
Answer the amount of information in bit that this memory system can record in one operation.

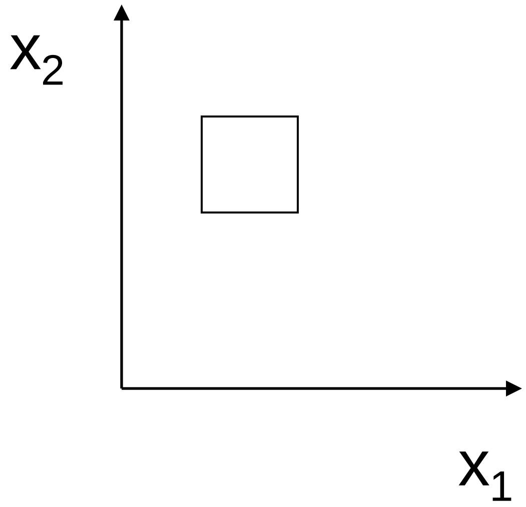
Example 3: Image measurement

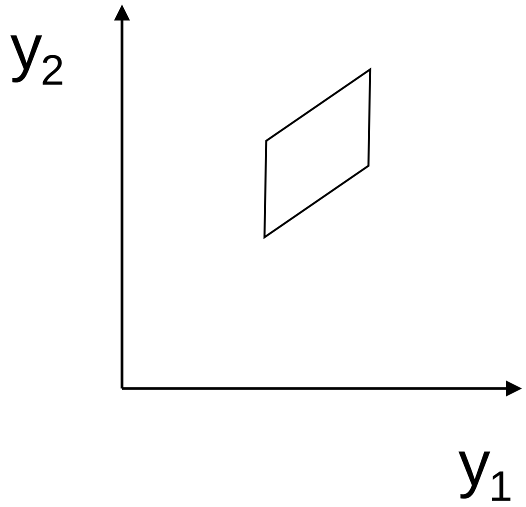
Relationship between the sensing accuracy of the displacement and the camera position

(x_1, x_2) : Object (point) position

y_i : Point position in the image of camera i



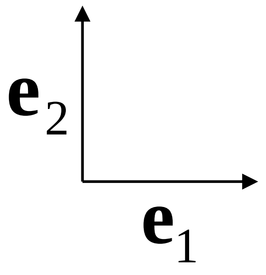


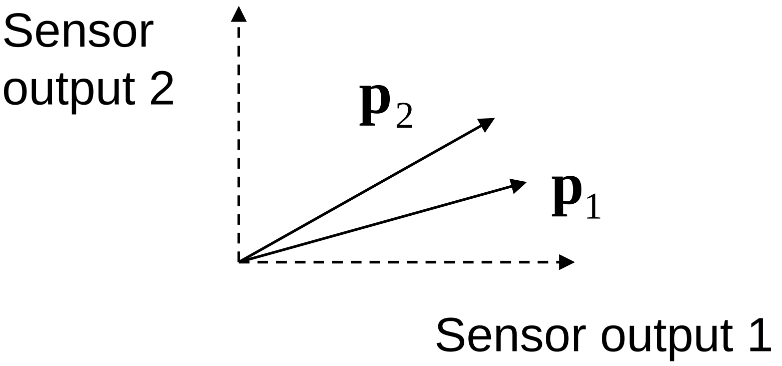


Example 4

The 300 Hz sine wave signal $p_{\mathrm{A}}(t)$ from a sound source A and the 120 Hz sine wave signal $p_{\mathrm{B}}(t)$ from a sound source B simultaneously reach a microphone, and $p = p_{\mathrm{A}} + p_{\mathrm{B}}$ is observed.
Obtain the error of amplitude estimation of p_{A} and p_{B} from the observed waveform of p.
Assume that the observation duration $T = 1\ \mathrm{s}$ and that the amplitude estimation errors when A and B exist independently are w_{A} and w_{B}, respectively.

Sensitivity vs direction

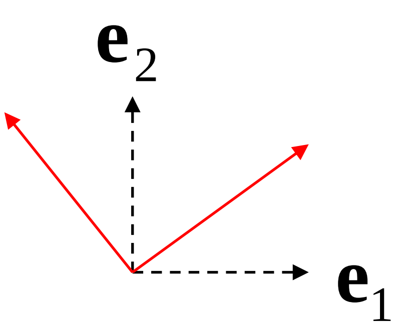


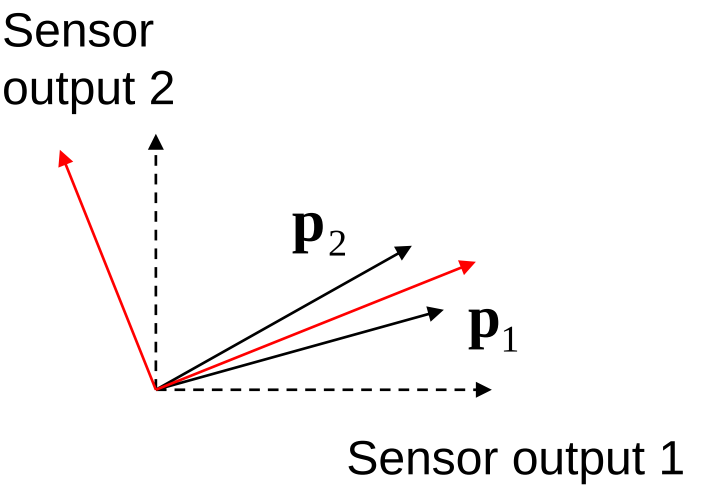


Space of $\boldsymbol{x}$

Space of sensor output $\boldsymbol{y}$

Understand the relationship between $\boldsymbol{x}$ and $\boldsymbol{y}$





Evaluation of measureability

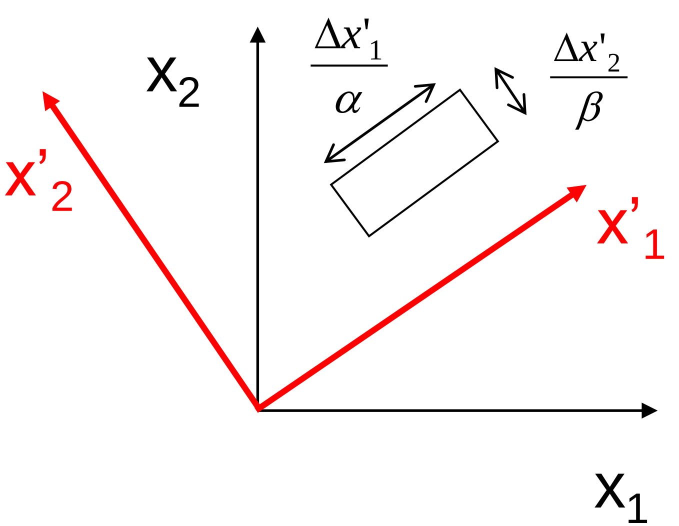
$$\mathbf{y} = A\mathbf{x}$$

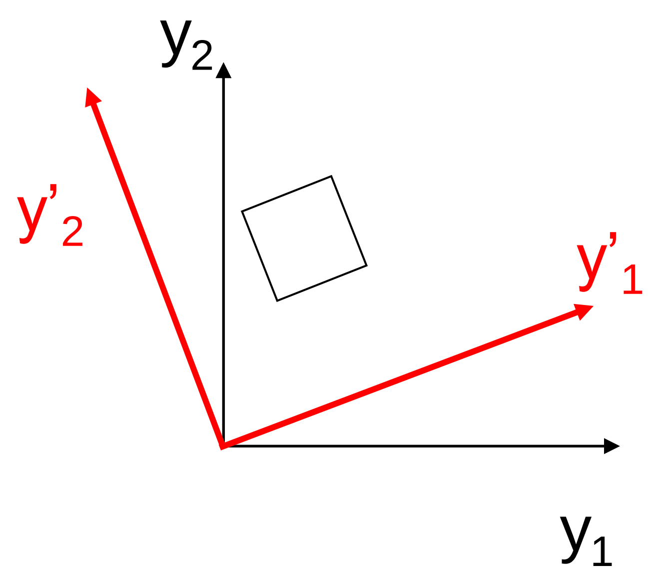
$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = R_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = R_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}$$

α^2, β^2: Eigenvalues of $A^{\mathrm{T}} A$





- Singular value decomposition

$$A = \begin{pmatrix} m \times n \end{pmatrix}$$

$$R_2 A R_1^{-1} = \begin{pmatrix} \text{m} \times \text{m} \end{pmatrix} \begin{pmatrix} \text{m} \times \text{n} \end{pmatrix} \begin{pmatrix} \text{n} \times \text{n} \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \\ 0 & & & \end{pmatrix}$$

$$\mathbf{y} = A\mathbf{x} \Rightarrow R_2 \mathbf{y} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \\ 0 & & & \end{pmatrix} R_1 \mathbf{x}$$

Deriving singular value decomposition

（1）$A^{\mathrm{T}}A$ is diagonalizable since it is Symmetric matrix

$$A^{\mathrm{T}}A = \begin{pmatrix} & n \times m & \\ & & \end{pmatrix} \begin{pmatrix} \\ m \times n \\ \\ \end{pmatrix}$$ = symmetric $n \times n$ matrix

$(m > n)$

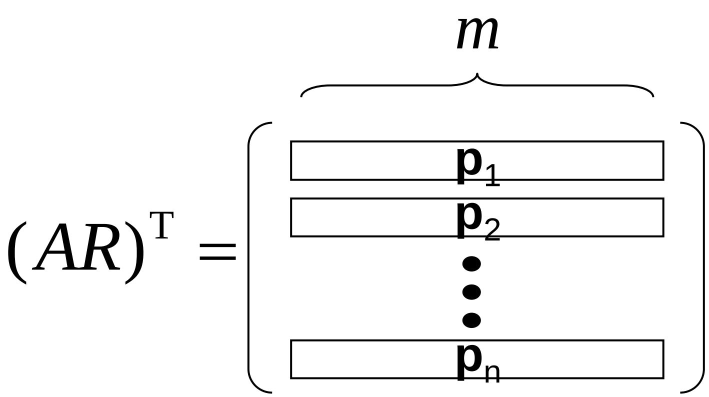
Therefore, we can find a orthogonal matrix R satisfying the following equation.

$$R^{\mathrm{T}}\left(A^{\mathrm{T}}A\right)R = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} = (AR)^{\mathrm{T}}(AR) \qquad \left(a_i > 0\right)$$

（2） Since

$$(AR)^{\mathrm{T}}(AR) = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix},$$

all the row vectors of $(AR)^{\mathrm{T}}$ are orthogonal to one another, and the length of each row vector satisfies $\|\mathbf{p}_i\| = \sqrt{a_i}$.

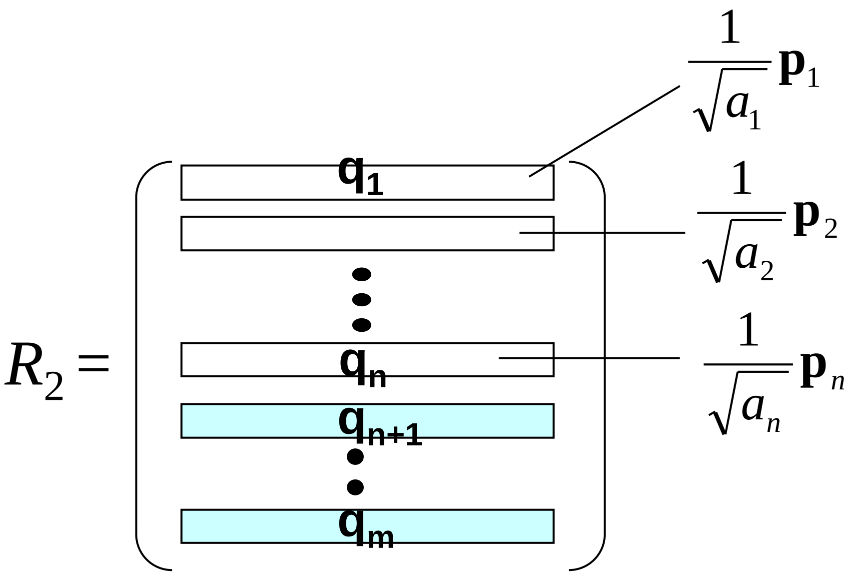


(3) Therefore, when the row vectors $\mathbf{q}_1 \sim \mathbf{q}_m$ in a orthogonal matrix R_2 safisfy

$$\mathbf{q}_i = \frac{\mathbf{p}_i}{\sqrt{a_i}}$$

for $1 \leq i \leq n \ \ (n < m)$,

R_2 satisfy the following equation.



$$R_2 A R = \begin{pmatrix} \sqrt{a_1} & & & 0 \\ & \sqrt{a_2} & & \\ & & \ddots & \\ & & & \sqrt{a_n} \\ 0 & & & \end{pmatrix}$$

(Singular value decomposition)

Understanding by coordinate transformation

Sensor output — Quantity to be measured

$$\mathbf{y} = A\mathbf{x}$$

coordinate transformation

$$\begin{cases} \mathbf{x}' = R_1 \mathbf{x} \\ \mathbf{y}' = R_2 \mathbf{y} \end{cases}$$

R_1 and R_2 satisfy

$$A = R_2^{-1} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \\ 0 & & & \end{pmatrix} R_1$$

$$\mathbf{y}' = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \\ 0 & & & \end{pmatrix} \mathbf{x}'$$

Proper coordinate transformation makes the non-diagonal component zero.

The measurement accuracy of $\boldsymbol{x}$' can be evaluated by the singular values

$$\boldsymbol{y}' = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \\ 0 & & & \end{pmatrix} \boldsymbol{x}' + \boldsymbol{w}'$$

Rotated sensor output

Rotated $\boldsymbol{x}$

① x_i for non-zero λ_i is measurable

When the SDs of the $\boldsymbol{w}$' components are comparable,

the measurement error of x_i is proportional to $\frac{1}{\lambda_i}$

② Values of λ_i are the same ⟺ A is orthogonal

⟺ Sensor outputs for x_i are orthogonal to one another

Conclusion of this chapter

Quantity to be measured: $\boldsymbol{x}$, Noise : $\boldsymbol{w}$,
Sensor output: $\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{w}$

∗ Each component of $\boldsymbol{w}$ is random and equal in variance.

In the above measurement system,
the ratio of the largest singular value to the smallest one of matrix A is close to 1.

⇕

Each component of $\boldsymbol{x}$ can be measured with equal accuracy.

Notes

- If the variances of the components of $\boldsymbol{w}$ are not equal,

  the methods in this chapter can be applied after scaling the parameters so that the variances are equalized.

- Note that this chapter's discussions assumed no correlations among the noise components.