Instrumentation and Information Processing

Chapter 3: Information in analogue pattern

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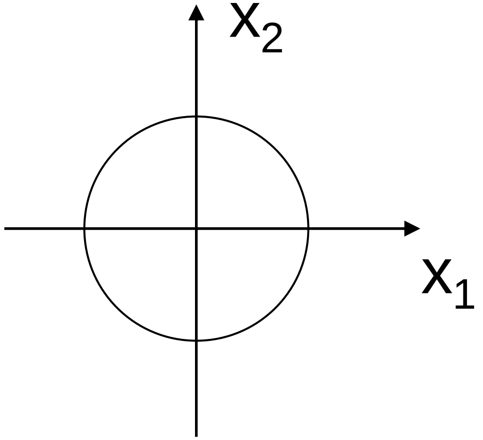
Amount of information recorded in an analogue pattern

First step:
Hypersphere in multidimensional space is written as follows.

2D $$x_1^2 + x_2^2 = R^2$$



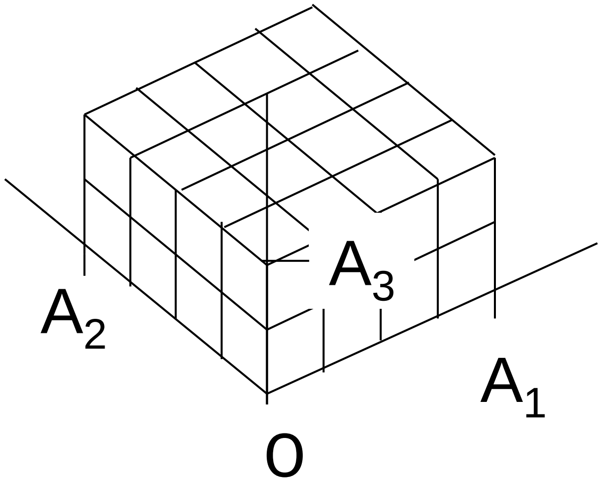
3D $$x_1^2 + x_2^2 + x_3^2 = R^2$$

n D $$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = R^2$$

- Signal is a point in a multidimensional space.
- It is possible to define "volume" in multidimensional space.

For example, a region V of n dimensional space

$$0 < x_i < A_i \qquad (i = 1, 2, \cdots n)$$



Includes a specific number of the unit hypercubes defined as

$$0 < x_i < 1 \qquad (i = 1, 2, \cdots n).$$

The number of the unit hypercube included in V is $A_1 \cdot A_2 \cdot \cdots A_n$.

The volume of a hypersphere

2D πr^2

3D $\frac{4\pi}{3} r^3$

nD $A r^n$

$$A(2m) = \frac{\pi^m}{m!}$$

Region of signal

$$|\mathbf{x}|^2 \equiv x_1^2 + x_2^2 + x_3^2 + \cdots + x_N^2 < S$$

U: Region of noise

$$U = \{\boldsymbol{x} \mid \ \|\boldsymbol{x}\|^2 < W\}$$

V: Region of signal + noise

$$V = \{\boldsymbol{x} \mid \ \|\boldsymbol{x}\|^2 < S + W\}$$

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3. Upper limit of transmittable amount of information

The volume of signal + noise

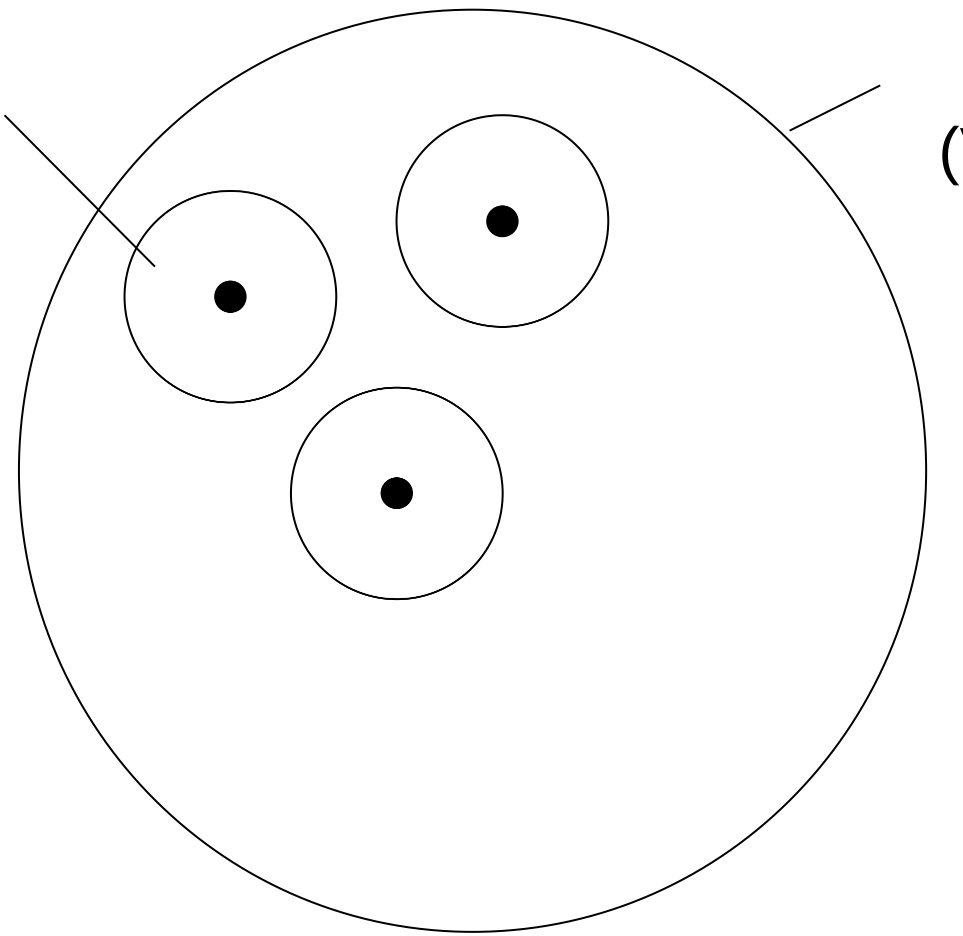
$$H = \log \frac{A(N)\sqrt{S+W}^{\,N}}{A(N)\sqrt{W}^{\,N}} = \log \sqrt{\frac{S+W}{W}}^{\,N} = \boxed{\frac{N}{2}\log\left(1+\frac{S}{W}\right)}$$ ビット

The volume of noise sphere

N: Number of data
S: Signal energy
W: White noise energy

The above H provides the upper limit. It has not yet been secured if actually H bit signal can be transmitted.

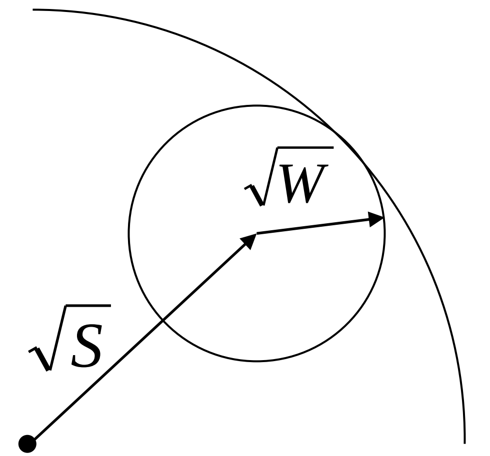
The upper limit of
the number of distinguishable statuses = V/U



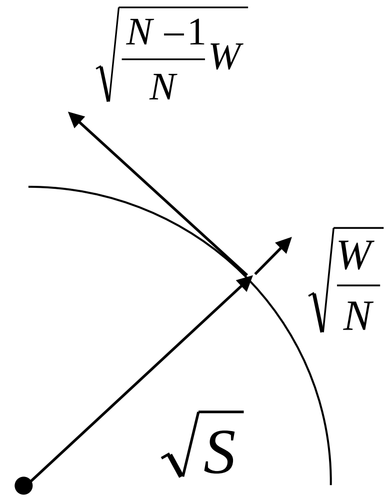
Region of “signal + noise”

In 2D or 3D case, the region of “signal + noise” is written as

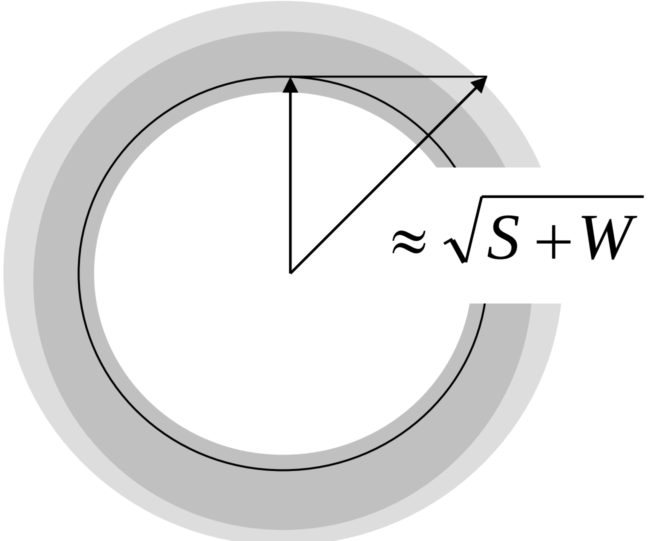
$$|\mathbf{x}|^2 < \left(\sqrt{S} + \sqrt{W}\right)^2$$



Region of "signal + noise" in multidimensional space



Most components of noise are orthogonal to signal



Supplementary explanation

Probability of $\|\boldsymbol{s}+\boldsymbol{w}\|^2 > S + W + \Delta$

$$\|\boldsymbol{s}+\boldsymbol{w}\|^2 = \sum_{i=1}^{N}\left(s_i^2 + 2s_i w_i + w_i^2\right)$$

$$= S + W + 2\sum_{i=1}^{N} s_i w_i$$

$$= S + W + 2\sqrt{S}w_s$$

w_s is the component included in ***w*** and parallel to ***s***, which is a probability variable that follows the normal distribution with variance W/N.

Therefore, most of $\boldsymbol{s}+\boldsymbol{w}$ exist in the region G between the two concentric spheres of radiuses $\sqrt{S+W+2\sqrt{SW/N}}$ and $\sqrt{S+W-2\sqrt{SW/N}}$.

The volume ratio r of the outer and inner spheres is given as

$$\log r = \log \frac{\left(S+W+2\sqrt{SW/N}\right)^{N/2}}{\left(S+W-2\sqrt{SW/N}\right)^{N/2}} = \frac{N}{2} \log \frac{S+W+2\sqrt{SW/N}}{S+W-2\sqrt{SW/N}}$$

When $N \to \infty$

$$\log r \to \infty, \quad \frac{1}{N} \log r \to 0 \, .$$

Therefore replacing V with G (above) does not change the value of $H/N \quad (N \to \infty)$.

4. Comparison of H with the previous chapter results

Case 1: $S \ll W$

$$H = \frac{N}{2}\log\left(1 + \frac{S}{W}\right) \approx \frac{1}{2\log_e 2}\frac{NS}{W} = 0.72\frac{NS}{W}$$

Case 2: $S \gg W$

$$H = \frac{N}{2}\log\left(1 + \frac{S}{W}\right) \approx \log\sqrt{\frac{S}{W}}^{N}$$

＊The results of ④ in the previous chapter is approximately equal to H.

④ Information transmission by combination of orthogonal signals

Slide in the previous chapter

Case 1: $S < W$

$$m = \frac{S}{\eta^2} = \frac{SN}{r^2 W} \quad \text{bits}$$

Case 2: $S > W$

$$\log_2 \left(\frac{1}{r} \sqrt{\frac{S}{W}} \right)^N = \frac{N}{2} \log_2 \frac{S}{r^2 W} \quad \text{bits}$$

In case 2: $S > W$

- Transmit the sum of N basis vectors (N-point waveforms) weighted by b_i as

$$s(n) = \sum_{i=1}^{N} b_i \phi_i(n)$$

- The maximum energy allocated to each basis vector is $\frac{S}{N}$.
- The maximum number of levels of b_i that can be identified without error is

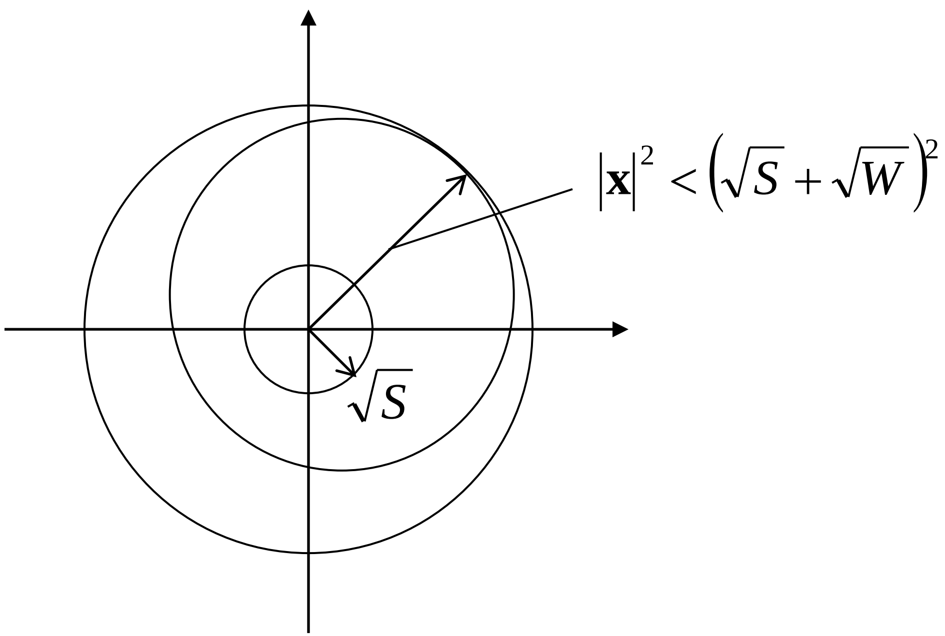
$$\frac{2\sqrt{S/N}}{2\sqrt{W/N}} = \sqrt{\frac{S}{W}}.$$

- Number of signal variations that can be transmitted without error:

$$\left(\sqrt{\frac{S}{W}}\right)^{N}.$$

5. Dividing multidimensional space

When considered in low dimensions, it seems that even one bit can not be transmitted if S <W.



Hypersphere 1

Things change as dimension get bigger

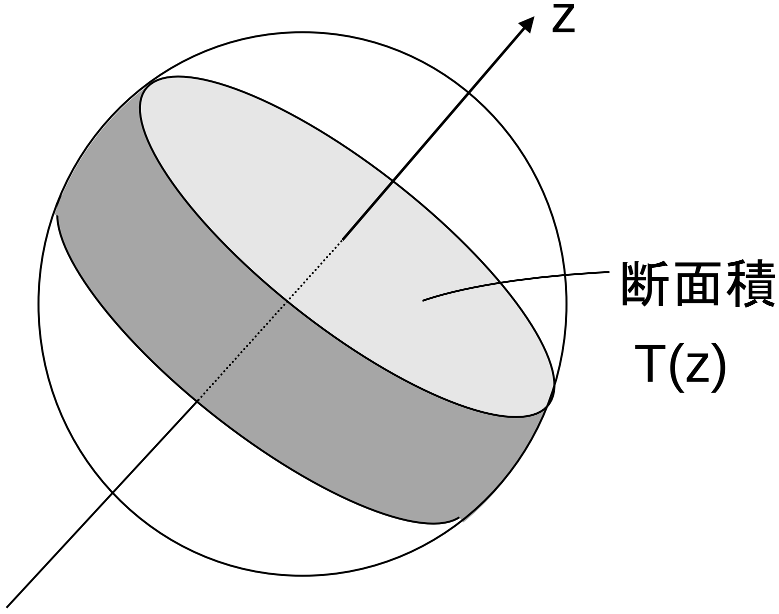
Most of the volume of the hypersphere is near the surface.

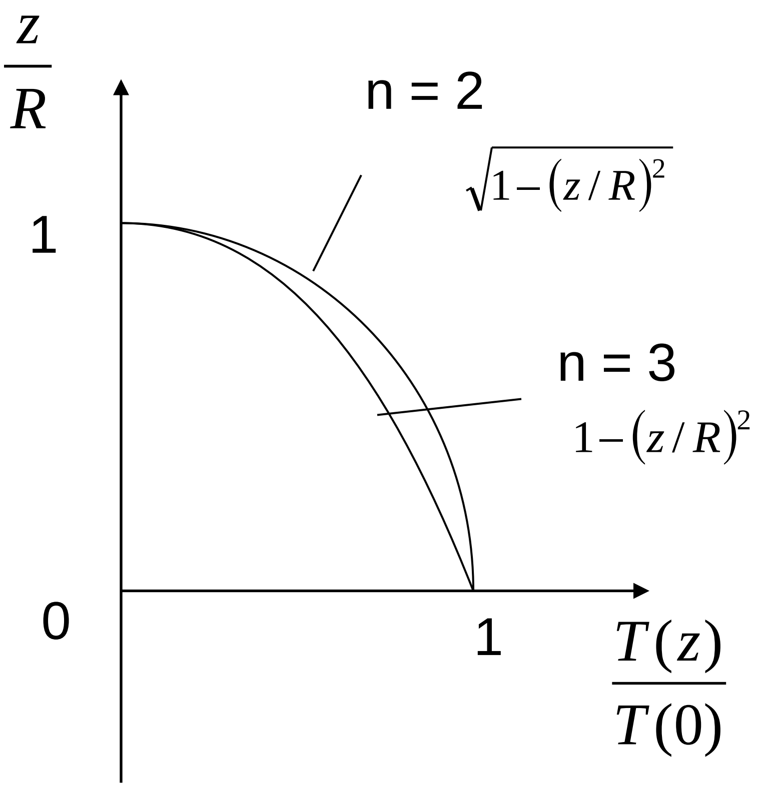
$$\frac{\text{Volume of sphere of radius } 0.99}{\text{Volume of sphere of radius } 1} = 0.99^n$$

$0.99^{300} = 0.05$

Hypersphere 2

Most of the volume is near the equator.



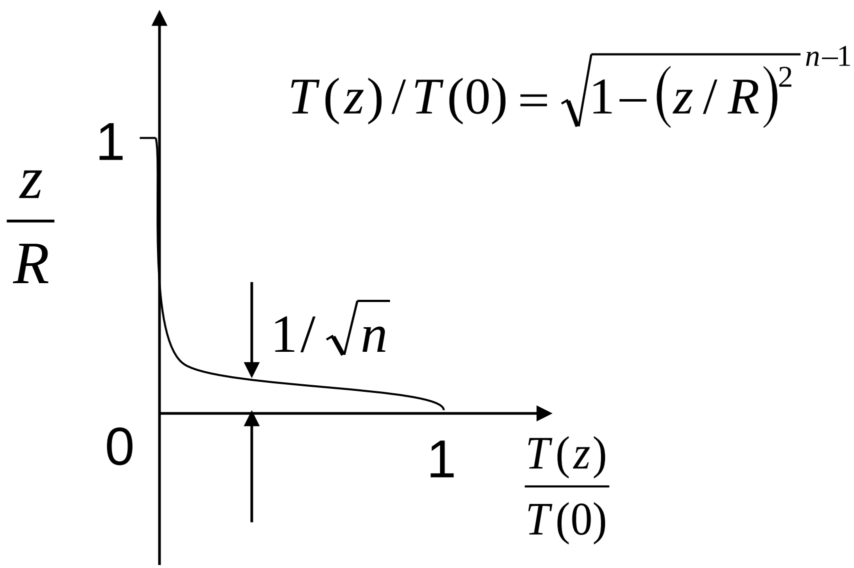


Cross-section of hypersphere

Cross section by n-1 dimensional hyperplane at distance z from origin

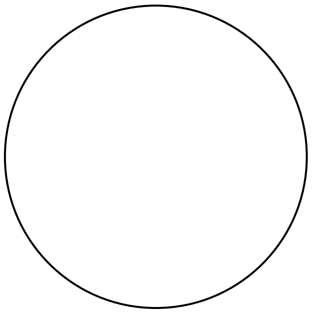
$=$ Volume of n-1 D hypersphere of radius $\sqrt{R^2 - z^2}$

$$= AR^{n-1}\sqrt{1 - z^2/R^2}^{\,n-1}$$



Sphere and cube

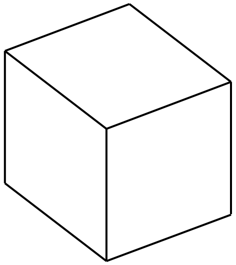
Sphere of radius R



$$V = \frac{\pi^{N/2}}{(N/2)!} R^N \quad (N\text{: 偶数})$$

$$\log V = N \log_e R - \frac{N}{2} \log_e \left(N / 2\pi e\right)$$

Cube of side length $\frac{R}{\sqrt{N/2\pi e}}$



$$V = \frac{R^N}{(N / 4\pi e)^{N/2}}$$

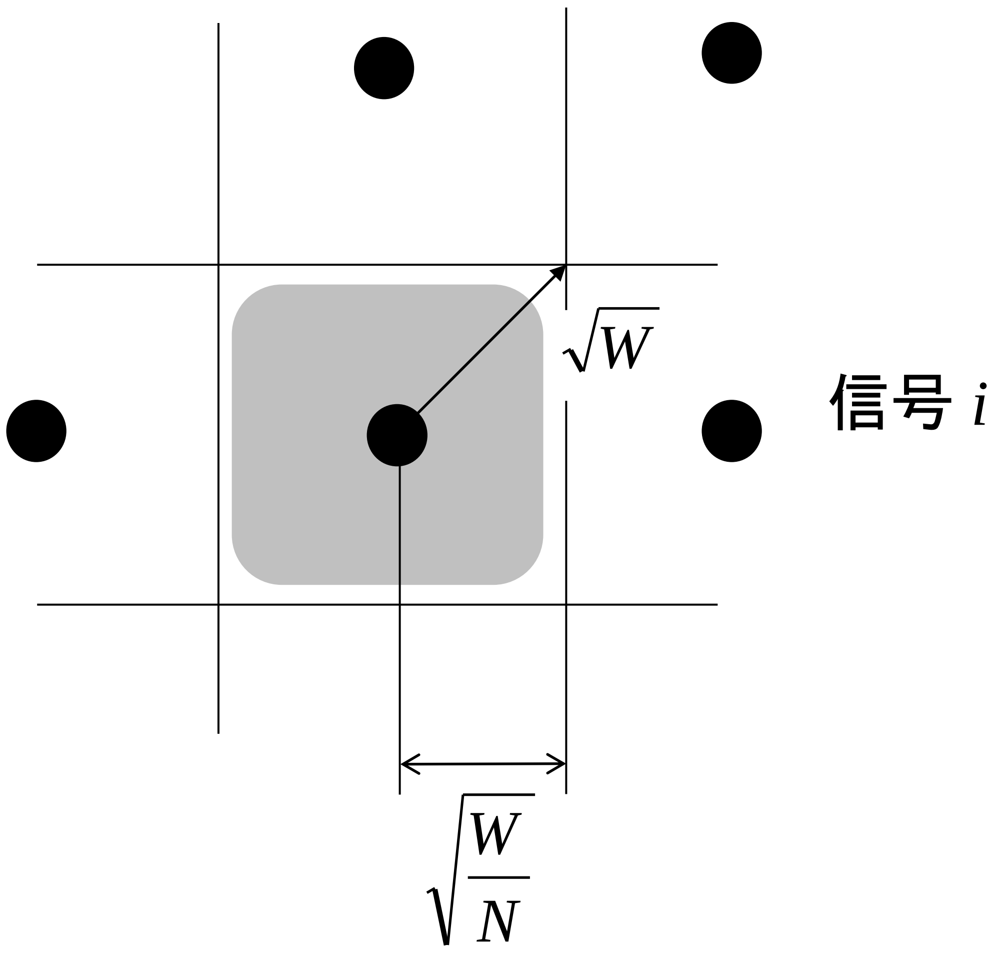
$$\log V = N \log_e R - \frac{N}{2} \log_e \left(N / 2\pi e\right)$$

⇔

Equal in the logarithm

6. Division by orthogonal grid

Even if the distance to the next signal is as close as $\sqrt{W/N}$, the overlapping volume of the noise sphere is small.



Volume of signal + noise region

$$\log \frac{A(N)\left(\sqrt{S+W}\right)^N}{\left(\alpha\sqrt{\frac{W}{N}}\right)^N} = \log \sqrt{\frac{S+W}{W}}^N = \boxed{\frac{N}{2}\log\left(1+\frac{S}{W}\right)}$$

Volume of cube of side length $\alpha\sqrt{\frac{W}{N}}$

If $\alpha = \sqrt{2\pi e} = 4.13$, it is equal to the volume of a sphere of radius $\sqrt{W}$.

Summary of this chapter

The amount of information that can be read from the pattern of $\boldsymbol{x}$ can be evaluated by the number of volume V where $\boldsymbol{x} + \boldsymbol{w}$ can move divided by the volume of the noise sphere.