# Instrumentation and Information Processing

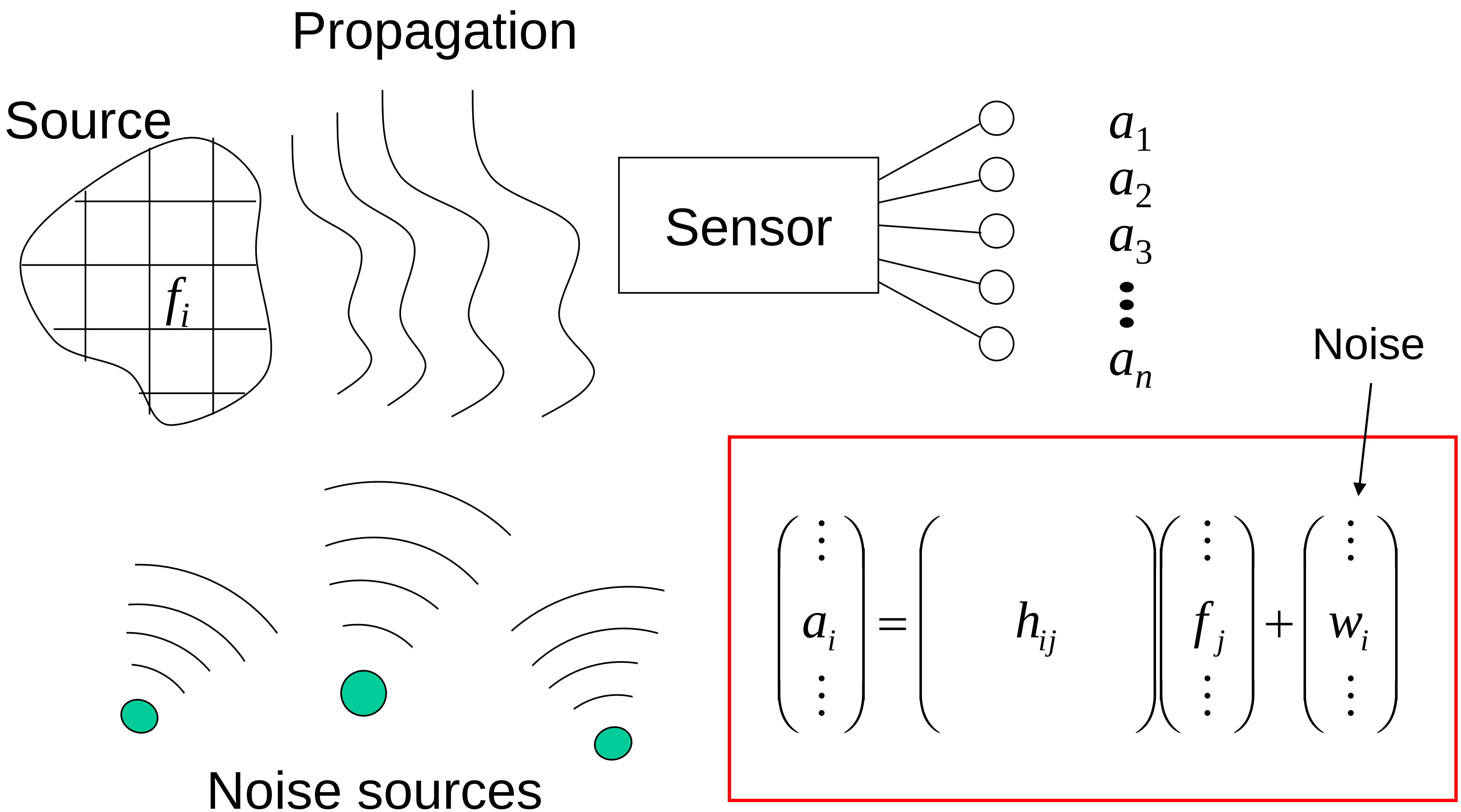
## Chapter 2:  Information transmission in noise

Hiroyuki Shinoda

https://hapislab.org/public/hiroyuki_shinoda/keisoku_joho

hiroyuki_shinoda@k.u-tokyo.ac.jp

# Pattern measurement in linear system




$$\begin{pmatrix} \vdots \\ a_i \\ \vdots \end{pmatrix} = \begin{pmatrix} & & \\ & h_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} \vdots \\ f_j \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ w_i \\ \vdots \end{pmatrix}$$

Obtain $\boldsymbol{f}$ from $\boldsymbol{a}$

# 1. Amount of Information

Recordable amount of information in a device is $n$ bit

The number of distinguishable states in a device is $2^n$

Computer memory, Hard disc, …
Paper book
Music CD
Analogue record ?

＊We assume there is a correspondence table between the device status and events.

# A measure on amount of information

$$I = \log_2(\text{Number of distinguishable statuses})$$

Why logarithm ?

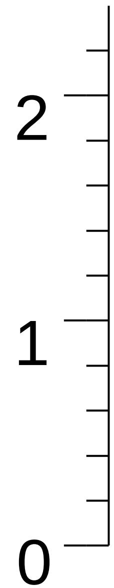
- proportional to the number or area of storage devices

# 2. Information transmitted in noise

## 1) Case of scalar measurement

Measure voltage V

Range: 0 ～ $V_0$ [V]



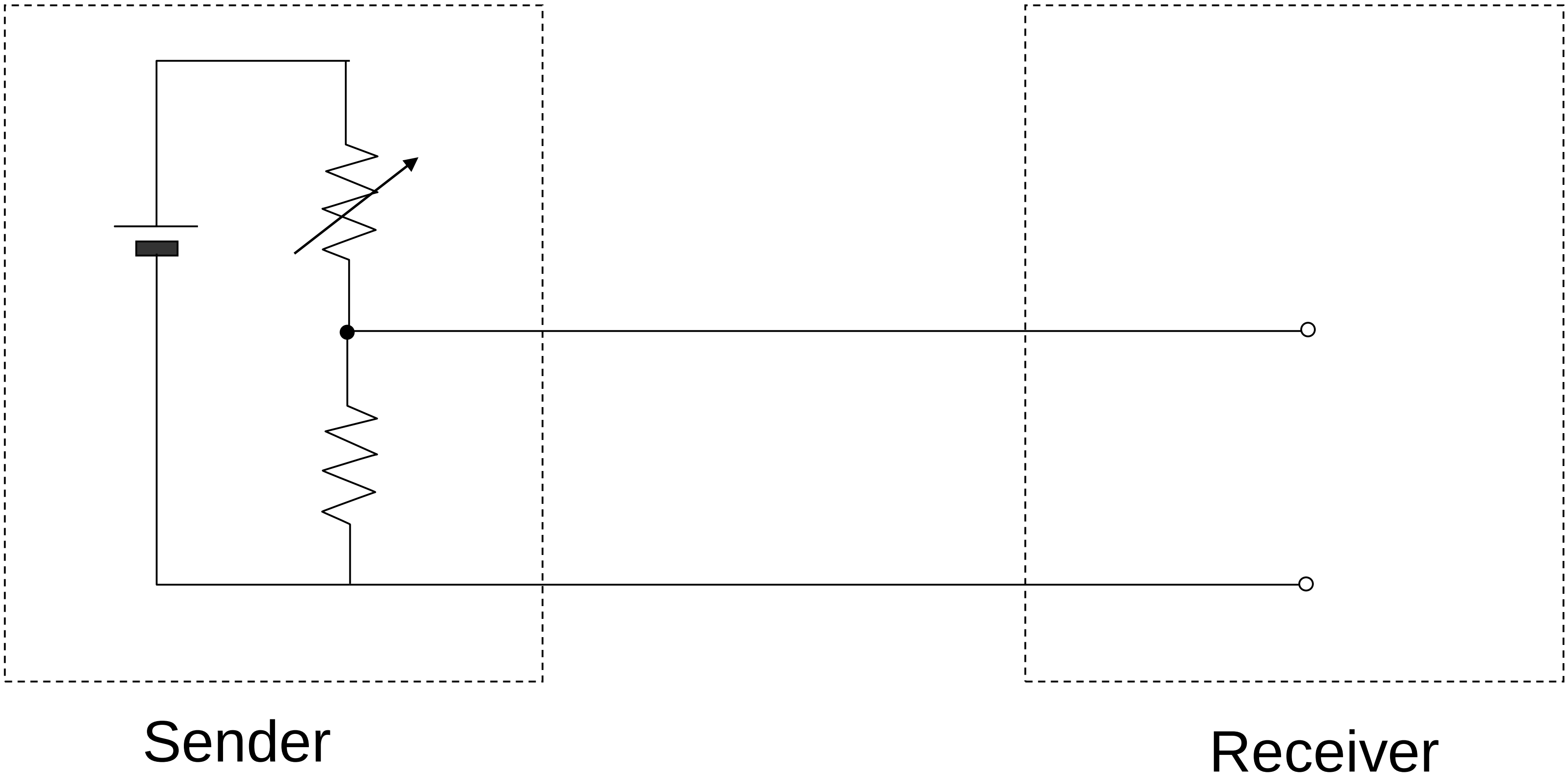

$I$: The number of distinguishable voltages $I$

$$I = \infty\ ?$$

# Quiz

## Communication by a single voltage

The receiver receives information by measuring the voltage the sender set.

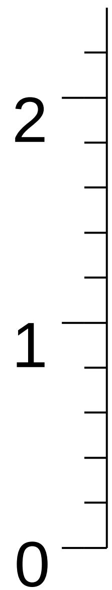


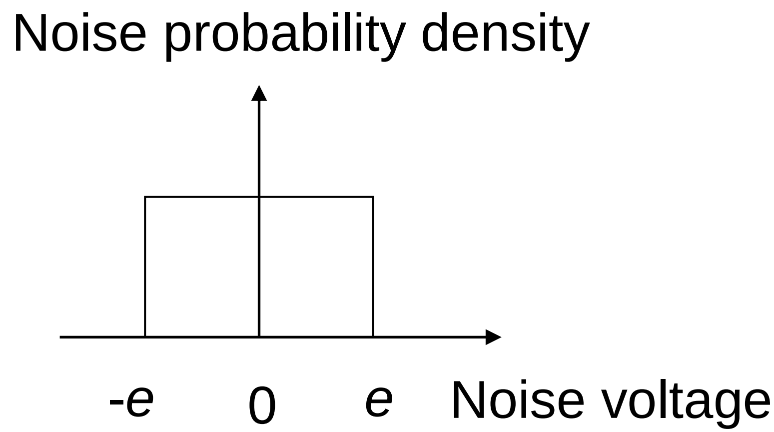

Infinite information is transferred in a moment ??

# 2.1 Scalar measurement with noise

Measure voltage V

Range: 0 ∼ $V_0$ [V]




Under the above noise, the maximum number of the distinguishable voltages by a single measurement

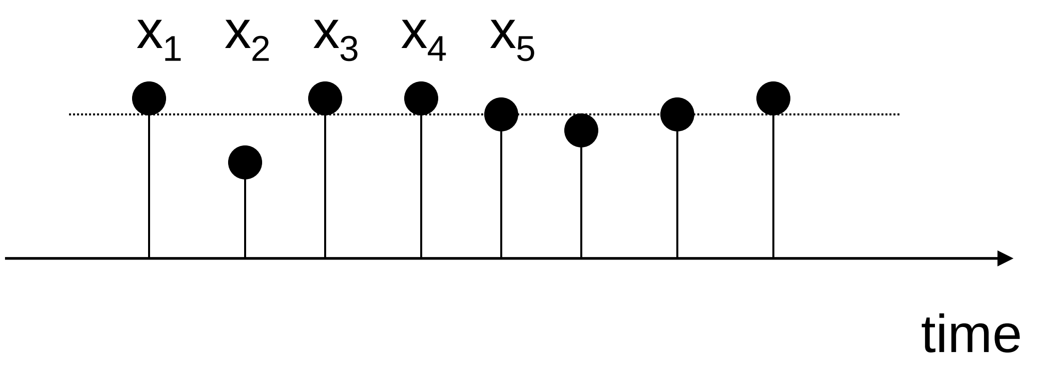
$$\frac{V_0}{2e}$$

# 2.1 Scalar measurement with noise

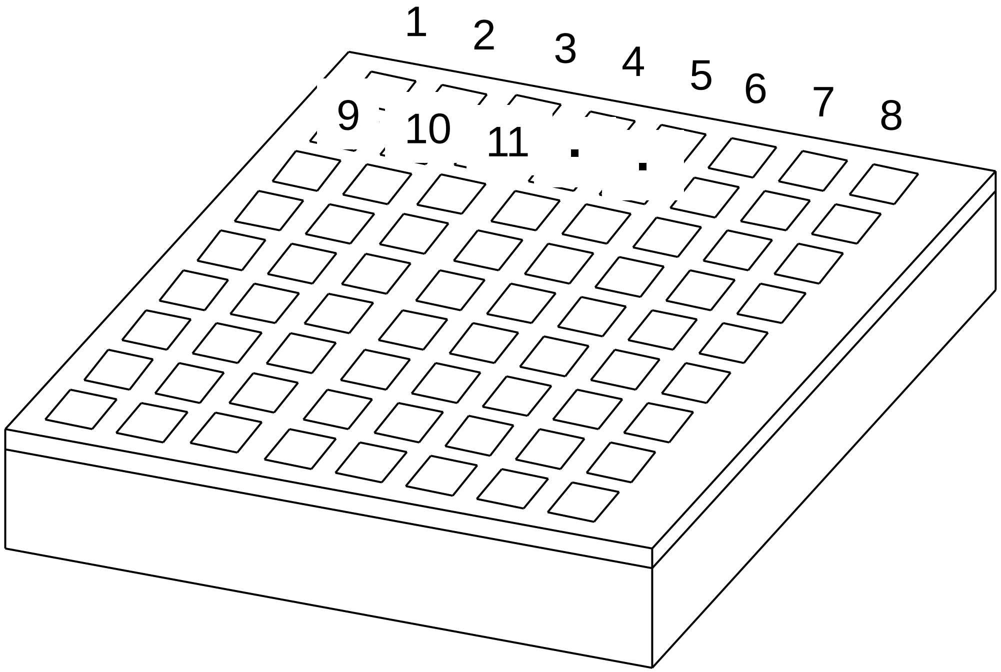
A potentiometer that can produce $\frac{V_0}{2e}$ distinguishable states is equivalent to a memory of $\log_2 \frac{V_0}{2e}$ bits.

# 2.2 Transmitted information when the receiver get the average

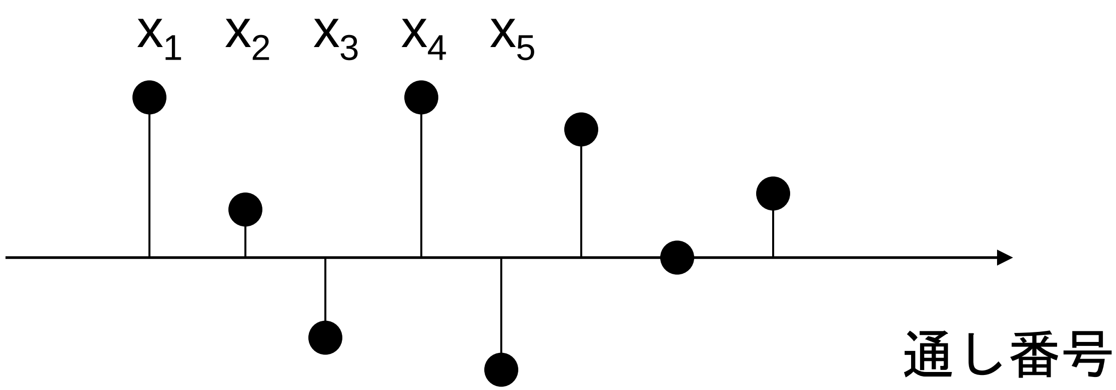
- Prior information：the signal voltage is constant
- $n$ time measured
- Noise is random and not correlated to signal




In that case, the number of distinguishable signal levels increases $\sqrt{n}$ times

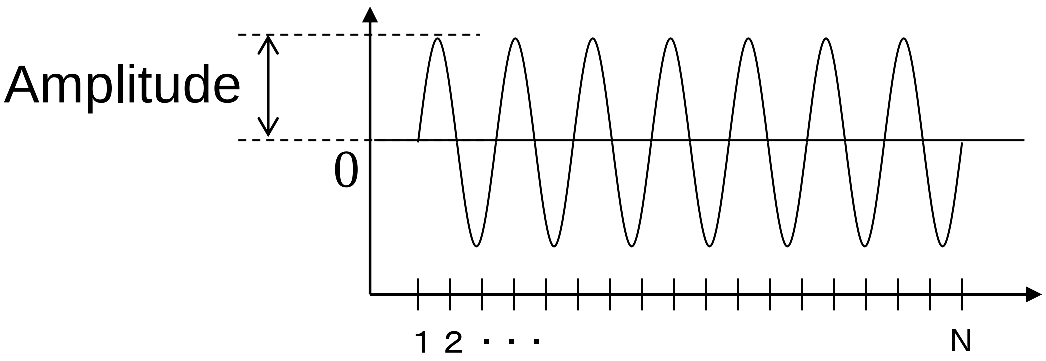



何についての平均か？
時間、空間、試行、・・・



# 3. Transmitted information amount vs. modulation method

## ① Information transmission by a single amplitude



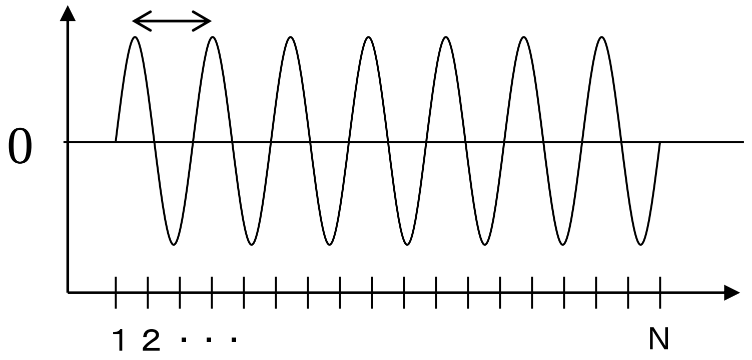

Scalar

$$s(n) = a\phi(n)$$

AM (Amplitude Modulation)

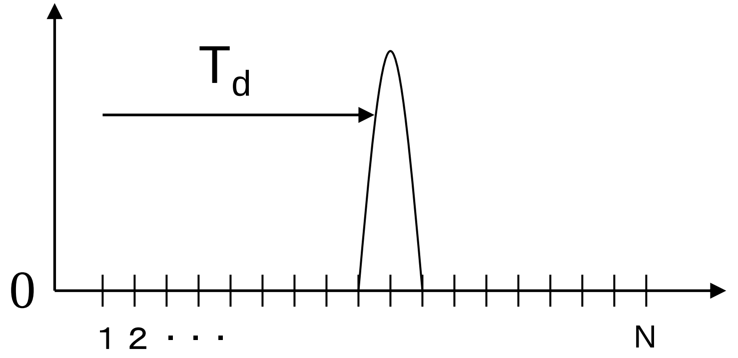
- $\phi(n)$: Not limited to a sinusoidal wave

## ② By frequency




FM変調
(Frequency Modulation)

## ③ By time shift



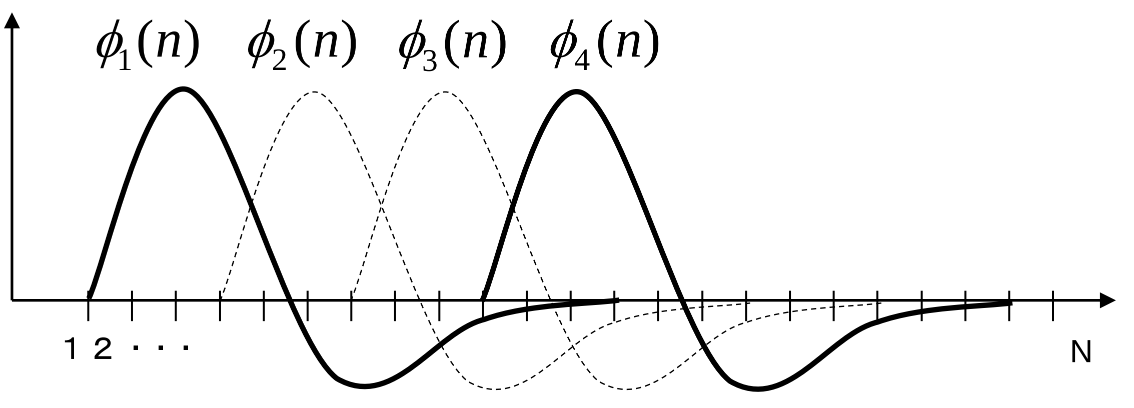

PIM
(Pulse Interval Modulation)

④ By combination of $m$ orthogonal vectors $\phi_i\,(n)$

Example: $(c_1, c_2, c_3, c_4) = (1,0,0,1)$

$$s(n) = c_1\phi_1(n) + c_2\phi_2(n) + c_3\phi_3(n) + c_4\phi_4(n)$$

$$= \quad \phi_1(n) \quad + \quad 0 \quad + \quad 0 \quad + \quad \phi_4(n)$$



# 3. Transmitted information amount vs. modulation method

How many bits are transmitted ？

Assumptions

[1] Signal length: $N$

[2] With a white noise uncorrelated to signal with energy $W$.

[3] The signal energy is smaller than $S$.

# ① Information transmission by a single amplitude

- The sent signal wave form: $s(n) = a\phi(n)$. The receiver obtains information by reading the scalar $a$.
- $\phi(n)$ is known by the receiver in advance.

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Consider the minimum difference of $a$ distinguishable by the receiver.

↓

Observed waveform $p(n) = s(n) + w(n)$

Signal (true value)

Noise

$p(n)$ is always possible to be decomposed as

$$p(n) = p_1\psi_1(n) + p_2\psi_2(n) + \cdots + p_N\psi_N(n)$$

Arbitrary selected orthonormal base

Choose $\varphi_1(n)$ so that
$\varphi_1(n) = \phi(n)$

Expected value of noise energy allocated to one component

= Expected value of noise energy allocated to a component parallel to $\psi_i(n)$ $(i = 1,2,\cdots,N)$

$$\overline{w_i^2} = \frac{W}{N}$$

- Noise is random and uncorrelated to signal
- If the probability distribution follows the normal distribution, the probability of

  $$|w_i| > 2.58\sqrt{W/N}$$

  is 1 %.

# Probability distribution of $w_i$

$w(n)$ is expanded orthogonally using an orthonormal base $\psi_1(n), \psi_2(n), \cdots, \psi_N(n)$ $\left( \sum_{n=1}^{N} \{\psi_i(n)\}^2 = 1 \right)$ as

$$w(n) = w_1\psi_1(n) + w_2\psi_2(n) + \cdots + w_N\psi_N(n).$$

Then, $w_i$ is given as

$$w_i = \sum_{n=1}^{N} w(n)\psi_i(n).$$

If the random variables $w(1), w(2), \cdots, w(N)$ have variance $\sigma^2$ the probability distribution of $w_i$ for a large $n$ forms a normal distribution with the variance

$$\sigma_i^2 = \sum_{n=1}^{N} \sigma^2\{\psi_i(n)\}^2 = \sigma^2 = \frac{W}{N}.$$

Maximum number of amplitude levels distinguishable in a white noise

$$I_{\mathrm{AM}} = \frac{2\sqrt{S}}{r \cdot 2\sqrt{W/N}} = \frac{1}{r}\sqrt{\frac{NS}{W}}$$

independent of the waveform of $\phi$.

∗ The $r$ is a comparable number to 1. The detail on how to determine the appropriate $r$ will be discussed later. For example, the probability of the reading error is 1 % for $r = 2.58$.

# ① Information transmission by a single amplitude

$$I_{\mathrm{AM}} = \log_2 N_{\mathrm{AM}} = \log_2 \sqrt{\frac{NS}{W}} \quad \text{bits}$$

（$r$ is omitted）

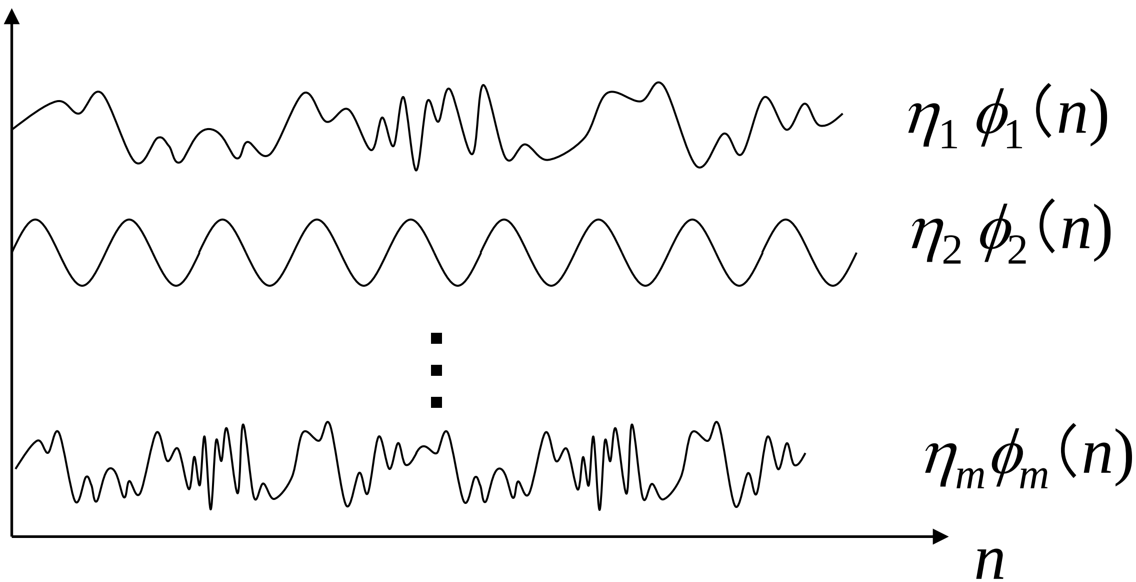
- $N$ point data is considered.
- A white noise with no correlation with the signal is assumed.

# ④ Information transmission by combination of orthogonal signals

The minimum signal energy for sending 1 bit information (two distinguishable states) in noise

$$\eta_i^2 \approx r^2 \frac{W}{N} \equiv \eta^2 \quad (\|\phi_i\| = 1,\ i = 1,2,\cdots)$$

- $r$ (comparable to 1) will be discussed later.



# Case 1: $S < W$

Prepare $m$ orthonormal vectors $\{\phi_1, \phi_2, \ldots, \phi_m\}$ and send the following signal representing a $m$-bit binary number

$$s(n) = a_1\eta\phi_1(n) + a_2\eta\phi_2(n) + \cdots + a_m\eta\phi_m(n) \qquad (\, a_i = 1 \text{ or } -1 \,)$$

$\eta$ is selected as the minimum value that can transmit 1-bit information, in order to maximize $m$.

# Case 2: $S > W$

Prepare $N$ dimensional orthonormal basis $\{\phi_1, \phi_2, \ldots, \phi_N\}$ and send the weighted sum of them. The weight of each basis vector $b_i$ is selected in the following range:

$$-\sqrt{\frac{S}{N}} < b_i < \sqrt{\frac{S}{N}} \ ,$$

then the signal

$$s(n) = \sum_{i=1}^{N} b_i \phi_i(n) \qquad (n = 1, 2, \ldots, N)$$

is transmitted. The receiver decompose the signal and obtain $b_i$. The difference (step size) of the levels of $b_i$ is determined as the minimum value that can be distinguished correctly in the noise.

- Learn about OFDM.

# ④ Information transmission by combination of orthogonal signals

Case 1: $S < W$

$$m = \frac{S}{\eta^2} = \frac{SN}{r^2 W} \quad \text{bits}$$

Case 2: $S > W$

$$\log_2 \left( \frac{1}{r} \sqrt{\frac{S}{W}} \right)^N = \frac{N}{2} \log_2 \frac{S}{r^2 W} \quad \text{bits}$$

Question

Consider the following strategy when $S < W$ instead of the previous slide one.
Obtain the amount of transmitted information and compare it with the previous result.

< Modified strategy >
Allocate more energy to limited number of $\phi_i$, that is, increase the maximum amplitude of $\eta\phi_i$ to $(2k+1)\,\eta\phi_i$ and set $2k$ states of amplitude for each of the basis. (Then, $m$ decreases.)

Since the maximum signal energy is fixed,
the maximum number $m$' of the basis is given as

$$m' = \frac{1}{(2k+1)^2} m$$

Therefore, transmitted amount of information is

$$\log_2 (2k)^{m'} = \frac{1}{(2k+1)^2} m \log_2(2\mathrm{k}) \quad \text{bits} \quad < m$$

# Question

Assume that an organism performs a kind of memory operation by autonomously changing and holding the concentrations of three chemical components $(x_1, x_2, x_3)$ [%] in the body fluid. In the body, there are three kinds of sensors that output as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and the sensory system can detect each component of $(y_1, y_2, y_3)$ with an error of about $\pm$ 0.1. Assume that $x_i$ can take values in the range of $0 < x_i < 10$, and there is no correlation among the sensing errors of each component.

Answer the amount of information in bit that this memory system can record in one operation.